Stable surfaces with constant anisotropic mean curvature and circular boundary

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Abstract

We show that, for an axially symmetric anisotropic surface energy, only stable disc-type surfaces with constant anisotropic mean curvature bounded by a circle which lies in a plane orthogonal to the rotation axis of the Wulff shape are rescalings of parts of the Wulff shape and the flat disc.

1 Introduction

We begin with a question. If we are given a variational problem for surfaces with boundary and the variational problem and the boundary of a critical surface admit the same symmetry, must the critical surface be symmetric? In [1] it was shown that, in the case where the functional is the area, any stable constant mean curvature immersion of a (topological) disc, which is bounded by a round circle, is necessarily axially symmetric and is hence a spherical cap or a flat disc. It is worth noting that earlier, the first author, [6], obtained the same conclusion under the assumption that the surface is an absolute minimizer of the volume constrained boundary value problem. Also, Kapouleas [5] has produced examples of higher genus constant mean curvature surfaces bounded by a circle, although little is known about their stability. For more than one boundary component, Patnaik [11] has produced a remarkable example of a non axially symmetric minimizer for the volume constrained Plateau problem where the boundary is prescribed to be two co-axial circles. In this paper, we obtain an extension of the result of [1] to the case of constant anisotropic mean curvature.

Let \( \gamma : S^2 \to \mathbb{R}^+ \) be a positive smooth function on the unit sphere \( S^2 \subset \mathbb{R}^3 \). We consider \( \gamma \) as an anisotropic surface density. This means that \( \gamma(\nu) \) gives the unit energy per unit area of a surface element having normal \( \nu \). The (anisotropic surface) energy of a surface \( \Sigma \) is thus

\[ F = \int_{\Sigma} \gamma(\nu) \, d\Sigma. \]

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There is a canonical closed convex surface associated with $F$, known as the Wulff shape which is defined by

$$W = \partial \bigcap_{n \in S^2} \{ Y \in \mathbb{R}^3 : Y \cdot n \leq \gamma(n) \}.$$ 

The surface $W$ is the absolute minimizer of $F$ among all closed surfaces which enclose the same three dimensional volume as $W$. In this paper, we will assume a convexity condition that $W$ is a smooth strictly convex surface. In particular, its curvature $K_W$ is everywhere positive.

Now let $X : \Sigma \to \mathbb{R}^3$ be a sufficiently smooth, oriented surface. If $X_\epsilon := X + \epsilon \dot{X} + \ldots$ is a compactly supported variation of $X$, then the first variation formula

$$\delta F := \partial_\epsilon F(X_\epsilon)_{\epsilon=0} = - \int_\Sigma \Lambda \dot{X} \cdot \nu \, d\Sigma$$

defines the anisotropic mean curvature $\Lambda$ (cf. [9]). The equation $\Lambda \equiv \text{constant}$ characterizes volume constrained equilibria of $F$.

A surface with constant anisotropic mean curvature is said to be stable if the second variation of the anisotropic surface energy $F$ is non-negative for all compactly supported variations of the surface which fix the enclosed oriented three-volume.

**Theorem 1.1** Let $F$ be a convex anisotropic energy with axially symmetric Wulff shape $W$. Denote by $D$ the unit disc in $\mathbb{R}^2$. Let $S^1$ be a round circle which lies in a plane orthogonal to the rotation axis of $W$. Let $X : (D, \partial D) \to (\mathbb{R}^3, S^1)$ be an immersion of a stable surface with constant anisotropic mean curvature. Then $X(D) \subset rW$ for some $r > 0$ or $X(D)$ is a flat disc.

## 2 Preliminaries

We assume that $\gamma(\nu)$ is a convex anisotropic energy density. Let $\chi : S^2 \to \mathbb{R}^3$ be the embedding such that $\chi(S^2) = W$ and $\chi^{-1}$ coincides with the Gauss map of $W$. If $\Sigma \to \mathbb{R}^3$ is an immersed surface with Gauss map $\nu : \Sigma \to S^2$, then $\xi = \chi \circ \nu$ is the Cahn-Hoffman field [3], which may be thought of as an anisotropic Gauss map. Since $T_{\xi(p)}W = T_p\Sigma$, we can consider $d\xi_p$ as a linear map of $T_p\Sigma$ to itself. Unlike the isotropic case, this map is not necessarily self-adjoint.

Let $\tilde{\gamma} : \mathbb{R}^3 - \{0\} \to \mathbb{R}^+$ denote the positive degree one homogeneous extension of $\gamma$, i.e. $\tilde{\gamma}(Y) = |Y| \gamma(|Y|/|Y|)$. The Cahn-Hoffman field $\xi$ can be computed by [2],

$$\xi_p = (\nabla \tilde{\gamma}(\nu))_p,$$

and the anisotropic mean curvature is given by

$$\Lambda = -(\text{div} \xi(\nu))_p.$$
We work locally on $\Sigma$ and choose a complex coordinate so that the induced metric has the form $ds^2 = e^{\varphi}|dz|^2$. We write
\begin{align*}
\xi_z = -\eta X_z - \beta e^{-\mu} X \bar{z}, \quad \bar{\xi}_z = -\bar{\eta} X \bar{z} - \bar{\beta} e^{-\mu} X \bar{z}
\end{align*}
(1)
and
\begin{align*}
\Xi := -d\xi \cdot dX =: 2\Re\left\{\frac{\beta}{2} dz^2 + \frac{\eta}{2} e^{\mu} d\bar{z} \bar{d}z\right\}.
\end{align*}
(2)
The quantity $\eta$ is called the complex anisotropic mean curvature ([7]) and it is given by
\begin{equation*}
\eta = \frac{\Lambda}{2} + \frac{\Gamma}{2}.
\end{equation*}
Here $\Gamma := \text{trace}_\Sigma (d\xi \circ J)$, where $J$ is the almost complex structure, i.e. $JX_z = iX_{\bar{z}}$. The form $\Xi$ is symmetric if and only if $\Gamma \equiv 0$. For example, $\Gamma \equiv 0$ always holds in the isotropic case and it holds if both $W$ and $\Sigma$ are axially symmetric with the same rotation axis. However, if $W$ is axially symmetric but not a sphere and $X$ is an immersion of a helicoid, then $\Lambda \equiv 0$ holds but $\Gamma$ is non zero on $\Sigma$, [8].

The points on $\Sigma$ where $d\xi_p = (\Lambda/2) dX_p$ are called anisotropic umbilics (A-umbilics).

**Lemma 2.1** Let $X : \Sigma \rightarrow \mathbb{R}^3$ be an immersion with constant anisotropic mean curvature. Then the following are equivalent:

(i) $p \in \Sigma$ is an A-umbilic.
(ii) $\beta(p) = 0$.
(iii) $(\Lambda^2 - 4K_\Sigma/K_W)(p) = 0$.

**Proof.** From (1), it is clear that a point $p \in \Sigma$ is an A-umbilic if and only if $\beta(p) = \Gamma(p) = 0$. However, below we will show that $\beta(p) = 0$ implies $\Gamma(p) = 0$. Therefore, (i) and (ii) are equivalent.

Let $\{e_j\}_{j=1}^2$ be a positively oriented orthonormal basis for the tangent space at $p$ which diagonalizes $d\chi_{\nu}(p)$, i.e. $d\chi_{\nu(p)}(e_j) = (1/\mu_j)e_j$. This is possible since $d\chi = D^2\gamma + \gamma I$ where $D^2\gamma$ denotes the Hessian of $\gamma$ on $S^2$. Note that $\mu_j$ are positive, because the Wulff shape is strictly convex. Let $(-\sigma_{ij})$ be the matrix representation of $d\nu_p$ with respect to this basis. It is straightforward to check that:

\begin{equation*}
\Lambda = \frac{\sigma_{11}}{\mu_1} + \frac{\sigma_{22}}{\mu_2}, \quad \Gamma = \sigma_{12}\left(\frac{-1}{\mu_1} + \frac{1}{\mu_2}\right), \quad K_\Sigma/K_W = \frac{\sigma_{11}\sigma_{22} - \sigma_{12}^2}{\mu_1\mu_2}.
\end{equation*}

If $z$ is a complex coordinate near $p$ with $z(p) = 0$, then there exists an angle $\theta$ such that at $z = 0$, $e^{-\mu/2} X_z = (1/2)e^{i\theta}(e_1 - ie_2)$. We compute at $p$:
\begin{align*}
-\frac{\beta}{2} e^{-\mu} &= e^{-\mu} d\xi(X_z) \cdot X_z \\
&= \frac{e^{2i\theta}}{4} d\chi d\nu (e_1 - ie_2) \cdot (e_1 - ie_2)
\end{align*}
\[
\begin{align*}
&= -e^{2i\theta} \left[ \frac{\sigma_{11}}{\mu_1} e_1 + \frac{\sigma_{12}}{\mu_2} e_2 - i \left( \frac{\sigma_{12}}{\mu_1} e_1 + \frac{\sigma_{22}}{\mu_2} e_2 \right) \right] \cdot (e_1 - ie_2) \\
&= -e^{2i\theta} \left( \frac{\sigma_{11}}{\mu_1} - \frac{\sigma_{22}}{\mu_2} - i \sigma_{12} \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right),
\end{align*}
\]

\[
4|\beta|^2 e^{-2\mu} = \left( \frac{\sigma_{11}}{\mu_1} - \frac{\sigma_{22}}{\mu_2} \right)^2 + \frac{\sigma_{12}^2}{\mu_1 \mu_2} + \frac{4\sigma_{12}^2}{\mu_1 \mu_2} + \frac{\sigma_{12}^2}{\mu_1} - \frac{1}{\mu_2}^2
\]

\[
= (\Lambda^2 - 4 \frac{K \Sigma}{K W} + \Gamma^2) .
\]

Since
\[
\Lambda^2 - 4 \frac{K \Sigma}{K W} = \left( \frac{\sigma_{11}}{\mu_1} - \frac{\sigma_{22}}{\mu_2} \right)^2 + \frac{4\sigma_{12}^2}{\mu_1 \mu_2} \geq 0
\]

holds, \( \beta(p) = 0 \) implies \( \Gamma(p) = 0 \). Moreover, one sees that (ii) and (iii) are equivalent. q.e.d.

**Proposition 2.1** Let \( X : \Sigma \to \mathbb{R}^3 \) be an immersion with constant anisotropic mean curvature. If the immersion is not totally anisotropic umbilic, then the anisotropic umbilic points are isolated. If \( p \) is an anisotropic umbilic and \( C \) is a sufficiently small closed curve around \( p \), then \( \text{Var}_C(\text{arg} \beta) := \text{the total variation of \text{arg} \beta over } C \) is equal to twice the negative of the winding number of the anisotropic principal direction fields around \( C \). In particular, \( \text{Var}_C(\text{arg} \beta) > 0 \) holds.

**Proof.** Let \( v := aX_z + \bar{a}X_{\bar{z}} \neq 0 \) be a tangent vector. We obtain from (1),
\[
d\xi(v) = -2\Re\{a\eta + \bar{a}\beta e^{-\mu}X_z\}, \quad Jv = 2\Re\{iaX_z\}.
\]

The condition for \( v \) to be an anisotropic principal direction is that \( d\xi v \cdot Jv = 0 \) holds. This is the same as
\[
\Re\{ia[\bar{a}\eta + a\beta e^{-\mu}]e^\mu\} = 0,
\]

which gives, using the definition of \( \eta \),
\[
|a|^2 \frac{\Gamma}{2} e^\mu - \Im\{a^2 \beta\} = 0.
\]

(This agrees with the well known condition \( \Im\{a^2 \Phi\} = 0 \) in the isotropic case, where \( \Phi \) is the Hopf differential.)

Let \( p \) be an A-umbilic and let \( C \) be a positively oriented curve around \( p \) which does not contain or pass through any other A-umbilic. This is possible since it was shown in [10] that the A-umbilic points are isolated. We assume
that $v$ now represents a vector in a continuous anisotropic direction field along $C$. Write $a = |a|e^{i\vartheta}$, $\beta = |\beta|e^{i\vartheta}$. We can write the previous equation as
\[
\sin(\vartheta + 2\theta) = \frac{\Gamma e^\mu}{2|\beta|} = \frac{\Gamma}{\sqrt{\Lambda^2 - 4(K_\Sigma/K_W)^2 + \Gamma^2}} < 1,
\]
by the lemma and the assumption that $C$ contains no A-umbilics. Note that the first equality in (4) is the same as
\[
\vartheta + 2\theta - \arcsin\left(\frac{\Gamma e^\mu}{2|\beta|}\right) = 0.
\]
However the last term on the left is a well defined continuous function along $C$, so its variation over $C$ vanishes and we get
\[
\text{Var}_C \vartheta = -2\text{Var}_C \theta.
\]
Since it was shown in [10] that the winding number of the direction fields around an A-umbilic is negative, this completes the proof. q.e.d

\[\text{Corollary 2.1} \text{ Let } \tilde{\Xi} \text{ denote the symmetrization of } \Xi, \text{ i.e. } \tilde{\Xi}(u,v) = (1/2)(\Xi(u,v) + \Xi(v,u), \text{ and let } T \text{ be an eigendirection of } \tilde{\Xi}. \text{ Then the singularities of } T \text{ are exactly the A-umbilic points and the winding number of } T \text{ around any A-umbilic is equal to } -(1/2)\text{Var}_C \arg \beta, \text{ where } C \text{ is as above.}
\]

\[\text{Proof. From (2) and (3), it is seen that the singularities of } T \text{ are exactly the A-umbilic points. The last statement is proved in the same way as the Proposition except that now the } \Gamma \text{ term is missing. q.e.d}
\]

\section{Proof of Theorem 1.1}

Let $X : (D, \partial D) \to (\mathbb{R}^3, S^1)$ be an immersion with constant anisotropic mean curvature $\Lambda$. If we consider a smooth variation field $\dot{X} = u\nu + T$ where $T$ is tangent to the immersion, then the pointwise variation of $\Lambda$ is given by, [9]
\[
\dot{\Lambda} = \frac{1}{2} J[u],
\]
where $J$ is the Jacobi operator of the immersion. This operator is given by
\[
J[u] = \text{div}[(D^2\gamma + \gamma I)\nabla u] + \langle(D^2\gamma + \gamma I) d\nu, d\nu\rangle u.
\]
The endomorphism $(D^2\gamma + \gamma I)$ is positive definite at each point; this is just the convexity condition for the Wulff shape $W$. It follows that $J$ is elliptic and self-adjoint.

The second variation of $\mathcal{F}$ for a volume-preserving variation which fixes the boundary with $X$ as variation vector field is $I[u] := -\int_D u J[u] \, d\Sigma$, where $d\Sigma$ is the area element of $X$. Denote by $\lambda_j$ the $j$th eigenvalue of the Dirichlet
eigenvalue problem for $J$. If $\lambda_2 < 0$, then a suitable linear combination $f$ of eigenfunctions of $\lambda_1$, $\lambda_2$ satisfies
\[
f|_{\partial D} = 0, \quad I[f] < 0, \quad \int_D f \, d\Sigma = 0.
\]
One obtain a volume-preserving variation of $X$ which fixes the boundary with variation vector field $f\nu$. Hence $X$ is unstable.

Proof of Theorem 1.1. The proof closely follows the proof of the result in [1]. As in [1] we consider the variation of $X$ given by
\[
X_\epsilon := \sigma_\epsilon X = X + \epsilon E_3 \times X + O(\epsilon^2), \quad E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \sigma_\epsilon = \begin{pmatrix} \cos \epsilon & -\sin \epsilon & 0 \\ \sin \epsilon & \cos \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
This is just a one parameter family of rotations with vertical axis applied to $X$. Therefore, if $\psi := E_3 \times X \cdot \nu$, then $J[\psi] = 0$ by (5) and $\psi|_{\partial D} \equiv 0$, since the boundary is set-wise fixed by the variation.

We will show that $\psi$ is identically zero. We first assume that $\psi$ is not identically zero, and we will show that $D \setminus \{\psi = 0\}$ has at least three components. (In fact if there are three components, then there must be four). If this is true, then $\psi$ is an eigenfunction belonging to some $\lambda_j = 0$ with $j \geq 3$ by the Courant’s Nodal Domain Theorem. Hence, $X$ is unstable by the above remark.

We compute
\[
\frac{1}{\mu_1} + \frac{1}{\mu_2} \partial_n \psi = 2e^{-\mu} \mathfrak{N}\{z^2\beta\}, \quad (6)
\]
where $n$ is the outward pointing unit normal along $\partial D$. The equality (6) will be proved in §4. It is enough to show that $\partial_n \psi$ has at least three zeros on $\partial D$, since at each of the zeros, a branch of the nodal set must enter into $D$ by the Hopf maximum principle ([4]). To do this, we consider $\text{Var}_{S^1}(\arg(z^2\beta)) = 4\pi + \text{Var}_{S^1}(\arg \beta)$. It is enough to show that $\text{Var}_{S^1}(\arg \beta) \geq 0$ holds, since then $\mathfrak{N}\{z^2\beta\}$ must have at least three zeros.

First assume that there are no A-umbilics on $\partial D$. By the corollary and general facts about indices of direction fields, $\text{Var}_{S^1}(\arg \beta)$ is equal to $-2$ times the sum of the indices of a direction field of $\tilde{\mathcal{E}}$ in $D$. This is clearly non negative.

The case where there are A-umbilics on $\partial D$ is handled by the usual type of indentation argument. For each A-umbilic point of the boundary, $\text{Var}_{S^1}(\arg \beta)$ is incremented by minus the winding number of a direction field of $\tilde{\mathcal{E}}$ around such a point, so again $\text{Var}_{S^1}(\arg \beta)$ is non negative.

Since the assumption that $\psi$ is not identically zero implies that the surface is unstable, we can conclude that $\psi \equiv 0$ holds. From this it follows that the immersion $X$ is axially symmetric. However, axially symmetric surfaces with constant anisotropic mean curvature are classified ([9]), and the only ones of disc type are either subsets of $rW$, in the case $\Lambda = -2/r$ or they are flat discs in the case $\Lambda = 0$. The former are known to be stable, in fact minimizing, by a result known as Winterbottom’s Theorem, [12], and it is easy to show that planer surfaces are also stable. q.e.d.
4 Appendix

We will prove (6). Let $z = x + iy$ be the usual coordinate in the disc and let $\zeta = \log z = \log r + i\theta$ in $D \setminus \{0\}$. Then using that $r \equiv 1$ on $\partial D$, we have

$$\frac{1}{2}(X_r - iX_\theta) = X_\zeta = zX_z, \quad \frac{1}{2}(X_r + iX_\theta) = X_{\bar{\zeta}} = \bar{z}X_{\bar{z}}.$$ 

If $n, t$ denote the unit conormal and tangent to $\partial D$, we have

$$n = e^{-\mu/2}X_r = e^{-\mu/2}(zX_z + \bar{z}X_{\bar{z}}), \quad t = e^{-\mu/2}X_\theta = e^{-\mu/2}(izX_z - i\bar{z}X_{\bar{z}}).$$

We then obtain from (1),

$$d\xi(t) \cdot n + d\xi(n) \cdot t = 2e^{-\mu} \Im\{z^2\beta\}. \quad (7)$$

Next note that since the surface is bounded by a circle, both $n$ and $t$ are principal directions of $W$. Specifically we have

$$d\chi(n) = \frac{1}{\mu_1} n, \quad d\chi(t) = \frac{1}{\mu_2} t,$$

Let $(-\sigma_{ij})$ denote the matrix representing $d\nu$ with respect to the basis $\{n, t\}$, we easily obtain, using $d\xi = d\chi \circ d\nu$,

$$d\xi(t) \cdot n + d\xi(n) \cdot t = -(\frac{1}{\mu_1} + \frac{1}{\mu_2}) \sigma_{12}. \quad (8)$$

Finally, we compute

$$\partial_n \psi = \partial_n(E_3 \times X \cdot \nu) = (E_3 \times n) \cdot \nu + (E_3 \times X) \cdot d\nu(n) = t \cdot d\nu(n) = -\sigma_{12}.$$

This with (7), (8) implies (6). \textbf{q.e.d.}

References


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